The Mathematics of the Gods and the Algorithms of Men

A Cultural History

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Introduction

About which reality does mathematics speak to us? It is widely supposed that mathematicians preoccupy themselves with abstract formulas, and that it is only for inexplicable reasons that these formulas have applications in every area of science.

We conceive of immaterial entities that seem subsequently to be destined to define models of phenomena that actually occur in the world. On the one hand, there are real, present things; on the other, mathematical concepts, creations of our mind which simulate their behaviour in a more or less effective way. Ignorance of the true reason behind the descriptive power of formulas and equations certainly doesn't help to clarify the underlying motivation behind mathematical thinking. It gives currency instead to the idea that mathematicians are not inclined to engage with the world. Mathematics continues to present itself as a science that elaborates ingenious operations with rules and concepts that seem to have been conjured up with the sole aim of their being executed correctly.[1] It matters little that some of these ideas have been suggested by the observation of natural phenomena; the operations rapidly produce advanced and complex concepts that distance themselves from observable reality and ultimately confirm the distorted image of mathematics as a merely linguistic game or a series of empty formulas.

If we turn to its remote history and to its deepest purpose, however, mathematics appears to be orientated very differently than is commonly supposed. Its origins allow us to understand that ancient arithmetic and geometry were beginning to assume the role not so much of describing or simulating real things as offering a foundation for the very reality of which they were a part. It was concrete things themselves – those that were directly and immediately perceivable – that were shifting and mutable, and therefore prone to appear unreal. To find precisely what removed them from such instability and evanescence, one had to look instead to numbers, to their relations and to the figures of geometry.

If we think about Zeno's famous paradoxes, the number-points of the Pythagoreans and the atomists of antiquity, about Plato's mathematical philosophy, the discovery of incommensurability and the significance of the concept of relationality (*lógos*), about Babylonian calculus and Vedic mathematics, we are faced with a great mass of knowledge designed to capture the most internal and invisible – as well as the most real – aspect of the things that exist in nature. The theory of numbers and of continuous mathematics elaborated in the nineteenth century presented itself as the ideal continuation of ancient Pythagoreanism, and of a vision of the world inspired by an intuition of the atomistic nature of reality. Mathematicians of the period, then, continued to contend that their symbolic constructions corresponded to very real entities, and the widespread impression was that on the success of their theories depended the foundation of knowledge that was necessary for understanding the world. When in the early twentieth century the principles of these theories became uncertain and began to undergo a critical revision, mathematics was obliged to search for the reasons that make a system of calculation genuinely concrete and reliable.

At this time a key term began to circulate persistently among mathematicians: *algorithm*, a word which denoted not so much an abstract formula as an actual process.[2] This process needed to unfold in a finite number of steps, from an initial set of data to a final outcome in space and time, according to the modalities predicted by a machine. The formal definition of 'algorithm', based on recursion, on Turing's machine or on other formulations, dates back to the fourth decade of the last century – but the first indications that it would be this concept of algorithm that would inherit the sense of mathematical reality – that is to say, all that mathematicians consider to be real and actual – were already being witnessed in the first decade of the twentieth century, in the early signs of mathematical intuitionism and in the first arguments with which the French mathematician Émile Borel confronted the semantic paradoxes and incipient crisis of fundamentals.

The science of algorithms followed a tumultuous arc of development that spanned the entire century, reaching a culmination in the formal definitions of the thirties, before bifurcating into two different but complementary trends after the construction of the first digital calculators: on the one hand, information theory, with its abstract notions of computability and of computational complexity; on the other, a science of calculus on a large scale, dedicated to resolving mathematical problems in physics, economics, engineering and computer science in purely arithmetical and numerical terms. The multiple philosophical facets of this second trend have not yet been sufficiently analysed, but it is already evident to all how much it has contributed in every area of life, to culture and social organization, with the multiplication of a diverse variety of calculation processes aimed at solving specific problems of the most varied kind.

In numerical calculation on a large scale the theoretical *effectiveness* of algorithms aims at achieving computational *efficiency*. And today it seems clear that, in order to be *real*, the very same mathematical entities that have been constructed through a process of calculation must be capable of being thought of in the same way as efficient algorithms. Today the efficiency depends above all on the way in which they facilitate growth in computational complexity and errors of calculation. In particular, the error depends on *how rapidly the numbers grow* in the course of calculation.

The reasons for the growth in the numbers are strictly mathematical and may be analysed thanks to relatively advanced theorems. But it is worth noting that the reason for this growth, in all of its aspects, was already the object of the closest scrutiny in ancient thought, and it is precisely the way in which the growth of quantities is treated in Greek geometry, in Vedic calculations and in Mesopotamian arithmetic that has contributed to the understanding of the causes of the growth of numbers in modern algorithms. The reason for this is as simple as it is surprising: some important *computational schemas* have remained unchanged since then, right up until the most complex strategies of which large-scale calculation avails itself today.

Where do these schemas derive from? In certain cases of particular relevance to modern science, the sources are clear: these schemas derive from a singular combination of human design and divine dictate. In Vedic India the altars of Agni, the fire god, had complex geometrical forms and needed to be capable of being enlarged a hundredfold without changing shape, by using specific techniques that can also be found in Greek geometry and Mesopotamian calculation. In Greece it so happened, as in the case of doubling the cube, that the enlargement of a form was also demanded by a deity. But the enlargement of the geometric form was in strict relation to the algorithms delegated to approximate those numbers which, when faced with having to measure geometric magnitudes such as the diagonal of a square, or the relation between a circumference and diameter, are irrational. It was the Vedic gods and those of Greece, long before the God of Descartes, who guaranteed a nexus between mysticism and nature, between our most intimate sphere and external reality. Mathematics offered at that stage, too, the principle of this possible connection. At any rate, the modalities of growth in ancient geometry, inspired by religious observance, are reflected today in the growth of numbers in digital calculations, having an essential impact on the stability of calculation itself and on the predictive power of mathematical models. In fact, the modalities of growth of *geometric* figures, in particular the square, are often correlated to *numerical* procedures that generate fractions p/q, which approximate irrational numbers, where p and q are whole numbers. But usually p and q grow much more rapidly the more rapid the convergence in method, with potential negative effects on the precision and stability of the entire process of calculation.

The thesis that irrational numbers are real entities, with an ontological status comparable to that of whole numbers, was an achievement of the mathematics of the end of the nineteenth century, and of the way in which the concept of the arithmetical continuum was defined at the time. But the development of the science of algorithms and digital calculation became in the twentieth century the expression of a new kind of opposition: a kind of final act of the perennial tension – already touched on in Zeno's paradoxes – between numbers and geometry, between the discrete and the continuous. It is legitimate to speak of a clash between them because the study of algorithms was propitiated from the very beginning of the twentieth century by a contest of ideas aimed at re-evaluating the most realistic and constructive aspects of mathematics, in contradistinction to those abstractions which had given rise to paradoxes and a crisis of fundamentals: on the one hand, the masterly command of Émile Borel which signalled the importance of defining mathematical entities through algorithmic constructions; on the other, the dramatic schism articulated by L. E. J. Brouwer and by mathematical intuitionism within the compendium of mathematics. Arguing that a number exists only if it is built, Brouwer issued a general challenge to the prevailing scientific system, calling into question the fundamental definitions of classical analysis.

The constructivist philosophies, based on the idea of actual calculability, have assigned new pre-eminence to that which seemed alien to the abstract vocation of mathematics; they have given importance, that is, to the concrete operation, the factuality of nature and, ultimately, to the computational process that develops inside a machine operating within the limits allowed by space and time. But it is equally evident that important computational strategies are modelled on the same schemas that mankind had elaborated during those eras when they were closely engaged in supposed communications with the gods. For ritual purposes, in Vedic India as in Greece, the reason for growth in magnitude was of fundamental importance and had to be confronted mathematically. And the basic blueprints for enlarging a geometric shape are still traceable in the most advanced computational mathematics. The schemas have not changed, though they have certainly been elaborated and perfected through complex mathematical theories. From these theories we also derive the rationale for their efficiency and their effective capacity to translate mathematical models of nature into pure digital information.

The same computational process, the same process articulated in a myriad of concrete automatic operations, can take place only thanks to abstract mathematical structures, inserted more or less artificially into the calculation. Mathematical abstraction is combined in a necessary and systematic way with the materiality of the automatic execution of operations. The calculation is made possible because of complex theoretical presuppositions and the special properties of numbers, functions and matrices.

The question thus remains open: are numbers real entities? And if we were to answer in the affirmative, are they all 'real' in exactly the same way? The two questions need to be tackled together. The history of the last century and an analysis of the concepts of number and algorithm allow us to see our way towards an initial conclusion: different kinds of numbers exist that do not have the same ontological status, but in relation to which it is possible to ascribe an existence in reality, for a variety of reasons and from different points of view. A key criterion for establishing the reality of numbers is the way in which they grow in the processes of calculation. And the first reason for this phenomenon should be sought in the analysis of geometrical growth elaborated in ancient thought, especially in Greek, Vedic and Mesopotamian mathematics.

1. Abstraction, Existence and Reality

Where does mathematics come from, and what are its objectives? Why are there triangles, squares, circles and pentagons? What kind of reality and existence may be attributed to numbers? Mathematics, as even some of the most intransigent formalists often admit, is real knowledge – and the object of this knowledge, we can say with confidence, is not arbitrary, does not depend on a capricious imagination or on the arbitrary choice of certain axioms or principles. Furthermore, it frequently happens to be perceived as an external reality, independent of the mind that elaborates it.

We usually think that mathematics is an abstract science, since in practice it extracts relationships and patterns from specific entities such as numbers in order to study common properties – as if the properties were in turn new entities that obey their own laws, with the advantage that all that may be said about the latter may be applied to the different specific entities from which the abstraction has been made. The reasoning, when it is abstract, thus becomes more general and more powerful. But abstraction makes more problematic the identification of an *essence* of mathematical entities, the location of an intrinsic character susceptible to definition. From their existence, at least as objects of our thought, a real essence of something that is stable and clearly recognizable does not seem to derive. The traditional reciprocal connection between *essentia* and *existentia* – whereby one term would not achieve the status of the real if the other were absent – seems to have been lost.[1] Does the essence of numbers consist in some kind of special nature that pertains to them which we can apprehend directly, or does it derive rather from the properties of an abstract domain of which the numbers in question are only a possible – not necessarily unique – example?

It is usually the second idea that is favoured. The properties that are studied separately form a system of truths derived from axioms – and mathematics, dissociated from any form of direct intuition, would then be considered as a science of pure formal relations, independent of every concrete interpretation of them. A specific numerical field,[2] such as that of real or complex numbers, satisfies the same axioms to which *other* mathematical fields also conform. It can thus happen that a number field may be identified only by an *isomorphism*, because it is indistinguishable from other mathematical entities that have the same properties. It's a circumstance that is enough to make mathematics completely impervious to the possible search for the specific nature of those numbers.

The recognition, or characterization, of a mathematical object usually depends on its specification in relation to properties that are independent of any possible construction or representation. And it is sometimes sufficient for a single, simple definition based on a few properties to identify an entire class of isomorphic domains.[3]

Are the instruments of logic suitable for establishing whether, and under what circumstances, a mathematical entity has real existence in some form or other, in an external world that is independent of us? There are conflicting views on this matter. There is, for instance, a significant difference between the views of Gottlob Frege and Bertrand Russell on this subject, to cite two of the pre-eminent exponents of *logicism*, which inspired the idea which sought to demonstrate that all mathematics is reducible to logic. For Frege, numbers were logical objects that one must define in some way. They are not created through definition: the definition merely demonstrates what exists in its own right.[4] Russell's position is different – decidedly more inclined towards *nominalism*, but not rigidly so. For Russell, logic as a whole is an indispensable instrument of knowledge of the *external* world. In *Our Knowledge of the External World* (1914), he wrote that the fundamental elements for explaining the nature of events are things, qualities and *relationships*, previously ignored by logic, which had assumed that all statements should have a subject–predicate form. Traditional logic had not taken into account the *reality* of relations, which were in fact indispensable to descriptions of the world and were needed in order to dissipate the errors of traditional metaphysics. In all likelihood, according to Russell, it was precisely traditional mysticism and metaphysics which had posited the idea of the unreality of the world that we perceive. Logic, Russell goes on to explain, serves to articulate a description of the world that originates with atomic propositions, which register the facts of empirical experience. From the atomic propositions we move to more complex ones, always thanks to logic. Logic obviously does not enter into the registration of elementary facts, but constitutes a comprehensive understanding, *a priori* in character, on which all potential deductions are based. This complex of deductions, while not deriving exclusively from sensory knowledge, should be considered to be knowledge that is real and effective. And mathematics, Russell concludes, is an essential part of it.

Nevertheless, if we pay attention to what logic actually tells us, at stages what seems to prevail is the most radical nominalism. Russell could not avoid acknowledging that the categories which define numbers are not things of which we have to establish the existence. Of course, with the language of logic, one strives to declare that a number, a set or a function exists. But the *reality* of numbers does not coincide at all with the existence established by this language. The logical propositions which assert that a mathematical entity exists serve 'not so as to succeed in knowing that which exists, but so as to know what a given assertion or given thesis, belonging to us or to others, *says* that there is; and this is properly speaking a linguistic problem, not an ontological one'.[5]

Logic alone does not suffice to establish an ontology of abstract objects. Nor should we be astonished by the words of Goodman and Quine, summarizing in 1947 the radical nominalism that was implicit in the attempt to base mathematics on logic:

> We do not believe in abstract objects. Nobody supposes that abstract entities – classes, relations, propositions, etc. – exist in space-time; but we want to say something more than this. We want to do away with their existence altogether.[6]

It is still worth specifying that 'abstract objects' are susceptible to existing in a variety of forms, and to becoming embodied in entities that are relatively concrete, with an existence in space and time. This circumstance depends on at least two distinct and different factors: the existence of an automatic calculation that develops in the physical space and time of a machine, and the widespread conviction that mathematical entities resemble living organisms, to the extent of being able to dictate the concrete conditions which permit us to study and understand them. The possibility of operating, subject to certain conditions, on numbers that exist physically as binary sequences in the memory of a calculator represents in itself a decisive fact. John von Neumann was able to observe how automatic calculation on a large scale, which had been progressing since the forties with the development of the first digital calculators, consisted of a calculation in *space* and *time* that was made possible by abstract structures that only pure mathematics was capable of elaborating. Mathematical abstraction and physical reality were inseparable in the new science of calculus.

When we talk about 'reality' we are straying into an extremely arduous terrain where things may be upended: a number may also be real that does not exist as we would wish or expect but that we somehow feel obliged to define in the course of meeting our needs or those of the algorithm.[7] In any case, Quine himself, in defining real numbers, was referring to preceding theories such as those of Dedekind and Weierstrass, which were in turn indebted to the various sources of knowledge stretching back *at least* to the theory of proportions in Book V of Euclid's *Elements*. And this theory was enhanced, in its own turn, by a knowledge of computation that was much more ancient than Euclidean theories. That which appears to be decisive in gauging the degree of reality is a range of experiences and situations, both theoretical and practical, that often have their origins in the remote past and which make it almost essential to arrive at a certain definition or posit a specific theory, that wants it to be configured in a certain way and not in another. Ernst Mach wrote passages of great interest on the way that scientific discoveries occur: one always ends up revisiting that which was already known, in an uninterrupted combination of the unknown and the known, of analysis and synthesis, of discovery and recognition. Discovery and invention are usually sustained by a preceding history, and by a chain of ideas that have facilitated them and rendered them necessary.

To simplify things, one might begin by assuming that a *realist* is someone who contends that the propositions of a scientific theory are either true or false, and that which makes them so is something external to ourselves, something different from sensory data, from our language and our thought.[8] So might it be plausible, then, to advance the notion that the reality of numbers depends on that of the physical world? The countless applications of mathematics to physics, economics, engineering, chemistry and information technology form a body of knowledge of impressive breadth and efficacy that seems capable of establishing the reality of formulas according to some kind of naturalistic philosophy, based on the idea that the universe consists exclusively of natural objects – objects situated in space-time and subject to causal laws. The universe is a vast book, wrote Galileo, in a famous passage in *The Assayer*, written in the language of mathematics and impossible to understand without resorting to triangles, circles and other types of geometrical form. But it is doubtful that this circumstance alone gives reality to mathematical entities; we can only affirm that it is not possible to be a realist with regard to physical theory and a nominalist with regard to the theory of mathematics.[9]

Therefore, it is not a given that a philosophical realism regarding physics is the only way of establishing a mathematical reality. However much it is our mind that elaborates the calculations, it is widely recognized that, if we put aside their applications to the real world, we find ourselves ultimately faced with algorithms, formulas and demonstrations that have more or less variable contours but are capable of dictating regardless of us the conditions and modalities of their existence, like wilful and obstinate creatures. This is why mathematicians so frequently claim that they feel obliged to recognize certain entities and not others, to be genuinely astonished that they seem to come from another world, or at least from a reality which is outside of our perceptual capacity, language and mental framework[10] – a circumstance which is enough in itself to make us reconsider Kant's thesis on the *a priori* character of mathematical thinking, according to which 'reason perceives only that which it produces itself according to its own design' (*Critique of Pure Reason*, Preface to the second edition, 1787). Mathematical entities, as conceived by many scientists and as they were perceived to be by Charles Hermite in particular, are not lifeless, artificial constructions but real, living beings with their own kind of coherence and intentionality, capable of guiding our research and determining the conditions of that freedom and autonomy that we customarily attribute to the dictates of our own reason.

'To define reality,' noted Simone Weil, 'nothing is more important than that.' And her own first conclusion was that: 'The real is transcendent; and this is Plato's essential idea.'[11] Hence it is necessary, for at least two reasons, to begin with the mathematics of antiquity: the question of the reality of mathematical entities has been around since the time of Pythagorean philosophy as well as Vedic and Mesopotamian calculations – and modern calculus, from the sixteenth century to the present day, has been based on constructions developed from ancient science. The question of the reality of numbers was posed in ancient Greece, at least implicitly, in attempts to understand what relations mathematical entities have with the infinite, with the non-being of the *ápeiron*, or infinite. Infinity was absence (*stéresis*), pure potentiality – and everything, to exist and to endure, had to oppose itself against the negativity of the limitless. This was, in Greek mathematics, the task of the *lógos*, of proportion, in which the precursors of modern numbers were found. The phenomenon of relation, and what derived from it, was an entity close to the gods. And not only in Greece. It was precisely in this *relation* that one needed to find the reality of numbers: not coincidentally, it was thanks to the reprising of concepts of relation and proportion (found in Euclid and in pre-Euclidean computational science) that the mathematicians of the nineteenth century established the ontology of numbers and the arithmetical continuum. It was this same circumstance that gave value to the logic of quantifiers through which we can affirm that a certain class *exists* (the existential quantifier), or that the conditions which allow a number to be identified must be satisfied for *every* value of certain variables (the universal quantifier) – a circumstance which allowed Bertrand Russell to assert that 'the sense of reality is vital in logic'.[12] This was not enough to really establish an ontology, but what was done subsequently to remedy such inadequacy is an ideal continuation of computational processes already well known in remote eras – in China, India and Mesopotamia as well as in Greece.

The processes of enumeration employed since the most ancient times often had the purpose of making things – both men and gods – real by means of a demonstrative or divine gesture, introducing them on to the stage of the world in order to render them actual and recognizable, with a full right to exist in space and time. Enumerations, censuses, lists and catalogues recur frequently in Homer and Hesiod, in Aeschylus and Herodotus, as well as in the Old Testament. Enumeration was a prerogative of the *lógos*, which suggested a process of selection and collection, of aggregation ordered by means of different entities into a single whole. Numbers themselves, in the Pythagorean tradition, were separated like a set of points in space, with an order similar to that of armies or of stars in the heavens. Consequently, the numbers with which one counted were not only abstract but also real and manifest. Nevertheless, with the demonstration that incommensurable quantities exist it was discovered that the natural number, the *arithmós*, was not enough to give actual existence to those entities that today we call irrational numbers. There are relations between magnitudes which cannot be expressed as relations between whole numbers. As Georg Cantor would go on to demonstrate in the nineteenth century with the *diagonal method*, phantasmal entities exist that escape from any kind of enumeration. Greek mathematics attempted to represent these entities indirectly, by means of a sequence of numbers, relations and geometrical figures regulated by a law of growth and decay. Modern mathematics would go on to use analogous sequences to define irrational numbers.

2. Mathematics of the Gods

It is hard to say why and where mathematics first arose. It is also pointless, perhaps, if we listen to the arguments of those who have sought to discourage any search for its origins by showing the intrinsic senselessness of such attempts. In *Human, All Too Human* (par. 249), Nietzsche denounces the weighing down, the deep weariness that can result from a gaze cast systematically towards the past. In the second of his *Untimely Meditations*, 'On the Uses and Disadvantages of History for Life', he argues that the soul of antiquarian man, who guards and archives the past in the most exacting way, can fall victim to the blind frenzy of a collecting mania and to an oppressive curiosity that acts as an obstacle to any impulse towards all that is new and vital. After Nietzsche, Michel Foucault explains that the origins of any system of knowledge are numerous and that it makes no sense to seek only one, as if it were the sole source of everything.

It seems to me that mathematics is hardly exempt from this rule: in any case, there are many kinds of mathematics – arithmetic, mathematical physics, algebra, geometry, analysis and statistics – that provide answers to different questions, obey a diversity of criteria and use various techniques of demonstration, even if in many cases the study of abstract mathematical structures has revealed surprising affinities between disparate domains, allowing us to glimpse the outlines of a unified knowledge. Over the course of time mathematical theories have often changed aspect, have been conceived and